Fractal set-valued measures

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Summary

Minkowski additive multimeasures
Minkowski sums and convex sets
Multimeasures
Spaces of multimeasures
IFS operators on multimeasures
Union additive multimeasures
Multimeasures
Spaces of multimeasures
IFS operators on multimeasures

Our aim is to investigate a form of self-similarity for set-valued objects, in particular set-valued measures (also called *multimeasures*). We will discuss two different types of "additivity" for these measures.

For each of these two types, we will discuss:

1. Definition and properties of multimeasures.
2. A complete metric space of multimeasures
3. Notions of self-similarity for multimeasures

Minkowski additive multimeasures

The first type of multimeasures we discuss are those which are additive with respect to Minkowski addition of sets:

Finite sum: $A + B = \{a + b : a \in A, b \in B\}$

Infinite sum: $\sum_n A_n = \{\sum_n a_n : a_n \in A_n, \sum_n \|a_n\| < \infty\}$

Simple properties of Minkowski addition

1. $A + B = B + A, \quad A + (B + C) = (A + B) + C.$
2. $A + \{0\} = A$

Thus we have a semigroup with identity (no inverses).

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4. A, B convex $\Rightarrow A + B$ is convex.
5. A, B compact $\Rightarrow A + B$ compact.
6. A, B, C compact and convex, $A + C = B + C \Rightarrow A = B$.

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5. A, B compact $\Rightarrow A + B$ compact.
6. A, B, C compact and convex, $A + C = B + C \Rightarrow A = B$.
7. A convex, $\sum_i \alpha_i = 1$ with $\alpha_i \geq 0 \Rightarrow \sum_i \alpha_i A = A$
8. A convex, $(\alpha + \beta)A = \alpha A + \beta A$ if $\alpha\beta \geq 0$.

Something to think about if you wish!

In fact, for any bounded $A \subset \mathbb{R}^d$, $\frac{1}{n} \sum_{i \leq n} A \to \text{co}(A)$ in the Hausdorff distance.

The support function of a convex set

Let $\mathcal{K}^d$ be the space of all non-empty compact and convex subsets of $\mathbb{R}^d$ and let $S^d = \{x \in \mathbb{R}^d : \|x\| = 1\}$.

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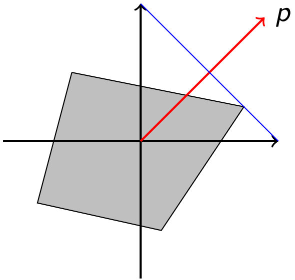
Given $K \in \mathcal{K}^d$ the *support function* of K is
$\text{spt} : \mathbb{R}^d \to \mathbb{R}$ given by $\text{spt}(p, K) = \sup_{\ell \in K} p \cdot \ell$.



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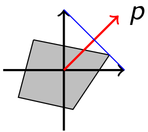

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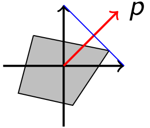

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$\operatorname{spt}(p, K)$ defines K, as p ranges over S^d, since

$$K = \bigcap_{p \in S^d} \{z \in \mathbb{R}^d : z \cdot p \leq \operatorname{spt}(p, K)\}.$$

The support function satisfies:

1. $\mathrm{spt}(p, K + L) = \mathrm{spt}(p, K) + \mathrm{spt}(p, L)$
2. $\mathrm{spt}(p, \lambda K) = \lambda\,\mathrm{spt}(p, K) = \mathrm{spt}(\lambda p, K)$ for $\lambda \geq 0$
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The Hausdorff distance between $K, L \in \mathcal{K}^d$ is given by

$$d_H(K, L) = \sup_{p \in S^d} |\text{spt}(p, K) - \text{spt}(p, L)|.$$

Multimeasures

A (Minkowski additive) multimeasure ϕ on $(\mathbb{X}, \mathcal{B})$ satisfies

$$\phi\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} \phi(A_i) \text{ for disjoint } A_i \in \mathcal{B}.$$

We assume $\phi(\emptyset) = \{0\}$ since otherwise $\phi(\emptyset)$ is unbounded (since $\phi(\emptyset) = \phi(\emptyset) + \cdots + \phi(\emptyset)$ for any number of terms).

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We will mainly be interested in multimeasures which take values in $\mathcal{K}^d$.

We assume $\mathbb{X}$ is compact and metric (compact for simplicity only).

Some general properties of multimeasures

Let ϕ be a multimeasure.

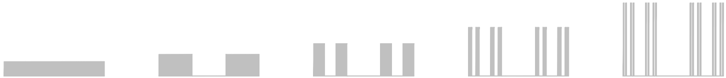
1. ϕ is *bounded* if $\phi(\mathbb{X})$ is bounded. This happens iff $\phi(A)$ is bounded for all A.
2. The *range* of ϕ is $\bigcup_{A\in\mathcal{B}} \phi(A)$ and is a bounded set iff ϕ is bounded.
3. for $p \in S^d$, $\phi^p(B) := \mathrm{spt}(p, \phi(B))$ defines a signed measure, a *scalarization* of ϕ.
4. $\bar{\phi}(B) := cl(\phi(B))$ and $\phi^*(B) := \mathrm{co}(\phi(B))$ are also multimeasures
5. If ϕ has no *atoms*, then $\phi(A)$ is convex for all A and the range of ϕ is also convex.

This last property is a generalization of Lyapunov's theorem for vector-valued measures.

Complete space of multimeasures

Our next task is to define a metric on multimeasures and find a suitable space of multimeasure which is complete under this metric.

The *Monge-Kantorovich* metric is often used for defining fractal probability measures because it transfers "geometric" contractivity of the underlying maps to the contractivity on measures.



Converging towards the "uniform" measure on the 1/3-Cantor set.

The Monge-Kantorovich metric on probabilities

For probability measures μ, ν, the Monge-Kantorovich metric is defined as $d_M(\mu, \nu) = \sup_{f \in \mathrm{Lip}_1(\mathbb{X})} \int_{\mathbb{X}} f \; d(\mu - \nu)$.

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This metric is related to mass transportation problems and can be thought of as the cost of moving μ to coincide with ν.

As an example, for two point masses δ_x, δ_y we have $d_M(\delta_x, \delta_y) = d(x, y)$.

Monge-Kantorovich on multimeasures

If ϕ, ψ are two Borel multimeasures on $\mathbb{X}$ with values in $\mathcal{K}^d$ we define

$$d_M(\phi, \psi) = \sup_{p \in S^d} d_M(\phi^p, \psi^p).$$

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Since ϕ^p, ψ^p are signed measures we must first define d_M on signed measures (the classical definition is only for probability measures).

Monge-Kantorovich for signed measures

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$f \in \mathrm{Lip}_1(\mathbb{X}) \Rightarrow f + c \in \mathrm{Lip}_1(\mathbb{X})$ and so $d_M(\mu,\nu) = +\infty$ if $\mu(\mathbb{X}) \neq \nu(\mathbb{X})$.

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The solution is to use $\mathcal{M}_{q,k} := \{\mu : \mu(\mathbb{X}) = q, \|\mu\| \leq k\}$, i.e. $\mu(A) \in [-k, k]$ for all $A \in \mathcal{B}$.

Notice that the two restrictions are automatically true for probability measures.

The MK metric is complete on $\mathcal{M}_{q,k}$

Theorem [La Torre, M] $(\mathcal{M}_{q,k}, d_M)$ is a complete metric space and d_M yields the weak* topology.

The restrictions for multimeasures

For a fixed $Q, K \in \mathcal{K}^d$ with $Q \subseteq K$, we define

$$\mathcal{M}_{Q,K} := \{\phi : \phi(\mathbb{X}) = Q, \phi(A) \subseteq K \text{ for all } A \in \mathcal{B}\}.$$

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In general, $\phi(A) \not\subseteq \phi(\mathbb{X})$ so $\phi(A) \not\subseteq Q$.

If $0 \in \phi(\mathbb{X} \setminus A)$ then
$\phi(A) = \{0\} + \phi(A) \subseteq \phi(\mathbb{X} \setminus A) + \phi(A) = \phi(\mathbb{X})$.

Thus if $0 \in \phi(A)$ for all A then $\phi(A) \subseteq \phi(\mathbb{X}) = Q$ and so we can use $K = Q$ (monotonicity).

restrictions for multimeasures continued

Let $\phi \in \mathcal{M}_{Q,K}$ and $p \in S^d$.

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Let $\phi \in \mathcal{M}_{Q,K}$ and $p \in S^d$.

$\phi^p(\mathbb{X}) = \text{spt}(p, \phi(\mathbb{X})) = \text{spt}(p, Q)$ and $\phi(A) \subseteq K$ so
$-\text{spt}(-p, K) \leq \text{spt}(p, \phi(A)) \leq \text{spt}(p, K)$.

Thus $\phi^p \in \mathcal{M}_{q,k}$ where $q = \text{spt}(p, Q)$ and for any
$k \geq \max\{|\text{spt}(-p, K)|, |\text{spt}(p, K)|\}$.

Thus the one choice of Q and K for the multimeasure ϕ gives a consistent choice of the spaces $M_{q,k}$ for the various scalarizations ϕ^p.

The MK metric is complete on $M_{Q,K}$

Theorem [La Torre, M] $(M_{Q,K}, d_M)$ is complete.

IFS operators on multimeasures

Our next task is to define a class of IFS type operators on $\mathcal{M}_{Q,K}$.

We copy the pattern from IFS on probability measures.

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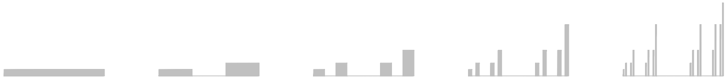
We copy the pattern from IFS on probability measures.

For probability measures, the standard operator is

$$T(\mu)(B) = \sum_i p_i \mu \circ w_i^{-1}(B),$$

p_i are probabilities and w_i are geometric contractions.



$w_1(x) = x/3$, $w_2(x) = x/3 + 2/3$, $p_1 = 1/3$, $p_2 = 2/3$.

We use $T(\phi)(B) = \sum_i \alpha_i \phi \circ w_i^{-1}(B)$.

The $w_i : \mathbb{X} \to \mathbb{X}$ are again the geometric contractions.

We take $\alpha_i : \mathbb{R}^d \to \mathbb{R}^d$ as linear.

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In order for $T : \mathcal{M}_{Q,K} \to \mathcal{M}_{Q,K}$ we need some conditions on the α_i:

- $\left(\sum_i \alpha_i\right) Q = Q$ (Mass preservation)
- $\left(\sum_i \alpha_i\right) K \subseteq K$ (Preservation of uniform bound)

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Countable additivity of $T(\phi)$ relies on the fact that each α_i is linear and thus continuous with respect to the Hausdorff metric.

Contractivity

Let s_i be the contractivity of w_i.

Theorem [La Torre, M] The IFS operator $T(\phi) = \sum_i \alpha_i \phi \circ w_i^{-1}$ is contractive if $\sum_i s_i \|\alpha_i\| < 1$.

Example 1: rectangles

The first example is simple: $w_1(x) = x/3, w_2(x) = x/3 + 2/3$

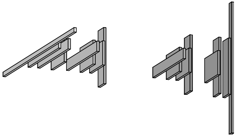
$$\alpha_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.7 \end{pmatrix}$$

$$K = Q = [-1, 1]^2$$

so $\alpha_1 + \alpha_2 = I$.

The invariant multimeasure is supported on the $1/3$ Cantor set:



Example 2: *zonotopes*

$Q \subset \mathbb{R}^d$ is a *zonotope* if $Q = \ell_1 + \ell_2 + \cdots + \ell_s$ where ℓ_i are closed line segments. We assume 0 is the midpoint of each ℓ_i.

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Let P_i be the orthogonal projection onto the span of ℓ_i and $c_i = |\ell_i|/|P_i Q|$ and $\alpha_i = c_i P_i$ so $\alpha_i(Q) = \ell_i$.

Let $w_j : \mathbb{X} \to \mathbb{X}$, $j = 1, \ldots, N$, have Lipschitz constant s_j and take $p_{i,j} \in [0,1]$ with $\sum_j p_{i,j} = 1$.

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Define T on $\mathcal{M}_{Q,Q}$ by $T(\phi) = \sum_{i,j} \alpha_{i,j}\phi \circ w_j^{-1}$.

Example 2: *zonotopes*

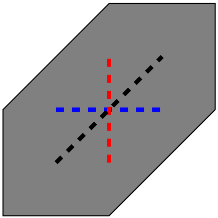
For an example., take $\ell_1 = [-1,1] \times \{0\}$, $\ell_2 = \{0\} \times [-1,1]$ and $\ell_3 = \{(x,x) : -1 \leq x \leq 1\}$.

Then Q is the convex polygon with vertices $\{(2,0),(2,2),(0,2),(-2,0),(-2,-2),(0,-2)\}$.

$$\alpha_{1,1} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{pmatrix} \quad \alpha_{1,2} = \begin{pmatrix} \frac{2}{6} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\alpha_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{6} \end{pmatrix} \quad \alpha_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$

$$\alpha_{3,1} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix} \quad \alpha_{3,2} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$



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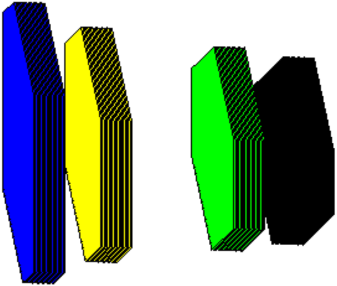
Example 2: *zonotopes*

The multimeasure can be illustrated as (this is just the second stage of the iteration, starting with the constant multimeasure Q):



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Next we discuss multimeasures which are additive with respect to the union operation. These are much simpler.

What is most interesting to me is the generality and range of examples one can construct.

Union additive multimeasures

The setting: Ω and $\mathbb{X}$ are two complete metric spaces,
$\mathcal{B}$ the Borel σ-algebra in Ω,
$\mathbb{H}(\mathbb{X})$ all non-empty compact subsets of $\mathbb{X}$.

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A (union additive) multimeasures ϕ on $\mathcal{B}$ satisfies $\phi(\emptyset) = \emptyset$,
$\phi(A) \in \mathbb{H}(\mathbb{X})$, for all $A \neq \emptyset$, and
$\phi(\bigcup_i A_i) = \overline{\bigcup_i \phi(A_i)} = \lim_n \bigcup_{i=1}^n \phi(A_i)$ (Hausdorff metric)

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It is an equivalent condition whether A_i are pairwise disjoint or not.

Union additive multimeasures

The setting: Ω and $\mathbb{X}$ are two complete metric spaces,
$\mathcal{B}$ the Borel σ-algebra in Ω,
$\mathbb{H}(\mathbb{X})$ all non-empty compact subsets of $\mathbb{X}$.

A (union additive) multimeasures ϕ on $\mathcal{B}$ satisfies $\phi(\emptyset) = \emptyset$,
$\phi(A) \in \mathbb{H}(\mathbb{X})$, for all $A \neq \emptyset$, and
$\phi(\bigcup_i A_i) = \overline{\bigcup_i \phi(A_i)} = \lim_n \bigcup_{i=1}^n \phi(A_i)$ (Hausdorff metric)

It is an equivalent condition whether A_i are pairwise disjoint or not.

Let $UA(\Omega, \mathbb{X})$ be the space of all such multimeasures.

Examples of union additive multimeasures

Take any $f : \Omega \to \mathbb{X}$ with $\overline{f(\Omega)}$ compact. Then $\phi(A) = \overline{f(A)}$ is a union additive multimeasure.

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Not all union additive multimeasures are of this form.

A simple example is $\Omega = [0,1] = \mathbb{X}$ with $\phi(\emptyset) = \emptyset$, $\phi(C) = \{1\}$ for countable C and $\phi(A) = [0,1]$ for any uncountable A.

Complete space of multimeasures

The simplest metric to place on $UA(\Omega, \mathbb{X})$ is

$$\hat{d}_H(\phi_1, \phi_2) = \sup_{\emptyset \neq A \in \mathcal{B}} d_H(\phi_1(A), \phi_2(A))$$

Theorem [La Torre, M] $(UA(\Omega, \mathbb{X}), \hat{d}_H)$ is a complete metric space.

IFS operators on multimeasures

Since the framework is so general, we can define a very general class of IFS operators on $UA(\Omega, \mathbb{X})$.

Let $w_i : \Omega \to \Omega, i = 1, \ldots, N$, map Borel sets to Borel sets.

Let $\alpha_i : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ be Lipschitz with factor s_i and such that

$$\alpha_i \left(\overline{\cup_n A_n} \right) = \overline{\cup_n \alpha_i(A_n)}.$$

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The condition on α_i can be met, for example, if $\alpha_i(A) = f(A)$ for some continuous $f : \mathbb{X} \to \mathbb{X}$.

We actually define two different operators. The issue is ensuring $M\phi(A) = \emptyset$ iff $A = \emptyset$.

First variant

For the first one, we assume that $\bigcup_i w_i(\Omega) = \Omega$. Thus $\emptyset \neq A \subseteq \Omega$ implies $w_i^{-1}(A) \neq \emptyset$ for some i.

$$M_1\phi(A) = \bigcup_{w_i^{-1}(A)\neq\emptyset} \alpha_i\left(\phi(w_i^{-1}(A))\right), \quad \emptyset \neq A \in \mathcal{B}.$$

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It is easy to see that $M_1\phi \in UA(\Omega, \mathbb{X})$ if $\phi \in UA(\Omega, \mathbb{X})$.

Second variant

For this one we make no additional assumptions but take a fixed $\Psi \in UA(\Omega, \mathbb{X})$ and define

$$M_2\phi(A) = \Psi(A) \cup \bigcup_{w^{-1}(A) \neq \emptyset} \alpha_i(\phi(w_i^{-1}(A))), \quad \emptyset \neq A \in \mathcal{B}.$$

Second variant

For this one we make no additional assumptions but take a fixed $\Psi \in UA(\Omega, \mathbb{X})$ and define

$$M_2\phi(A) = \Psi(A) \cup \bigcup_{w^{-1}(A) \neq \emptyset} \alpha_i(\phi(w_i^{-1}(A))), \quad \emptyset \neq A \in \mathcal{B}.$$

Again it is easy to see that $M_2\phi \in UA(\Omega, \mathbb{X})$ if $\phi \in UA(\Omega, \mathbb{X})$.

Contractivity

Theorem [La Torre, M] Let $s = \max_i s_i$ be the maximum of the Lipschitz factors of the α_is. Then for $j = 1, 2$

$$\hat{d}_H(M_j\phi_1, M_j\phi_2) \leq s\hat{d}_H(\phi_1, \phi_2).$$

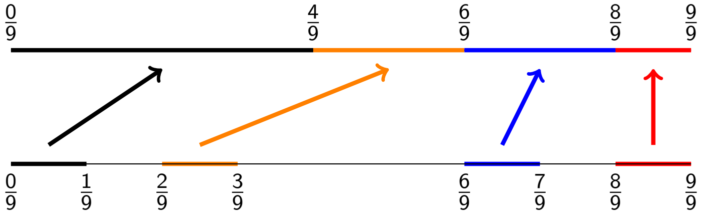
Examples

The first example is one where the values of the measure are subsets of the Sierpinski triangle.

Let $\Omega = [0,1]$, $\mathbb{X} = [0,1]^2$, $w_i(x) = x/3 + i/3$ for $i = 0, 1, 2$ and $\alpha_0(x,y) = (x/2, y/2)$, $\alpha_1(x,y) + (x/2 + 1/2, y/2)$ and $\alpha_3(x,y) = (x/2, y/2 + 1/2)$.

Examples: intervals as "probability"

We take $\Omega = [0,1] = \mathbb{X}$, $w_1(x) = x/3$, $w_2(x) = x/3 + 2/3$.
Fix $p \in (0,1)$, set $\alpha_1(x) = px$ and $\alpha_2(x) = (1-p)x + p$.



$p = 2/3$

Fractal recursive partitions

Fractal multimeasures

F. Mendivil, (Joint work with D. La Torre)

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These two examples are driven by a process which recursively generates a partition.

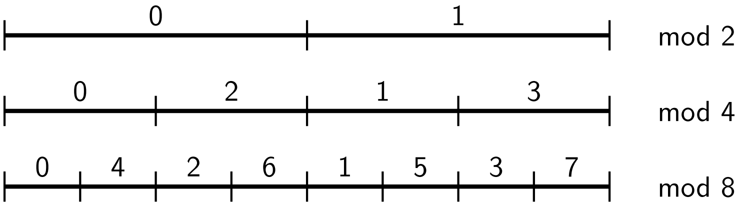
An IFS with probabilities also does this, in the sense that it partitions the total probability, $[0, 1]$, recursively.

Modular 2^n classes

$\Omega = [0,1]$, $\mathbb{X} = \mathbb{N} \cup \{0\}$, $w_0(x) = x/2$, $w_1(x) = x/2 + 1/2$
$\alpha_0(n) = 2n$, $\alpha_1(n) = 2n + 1$.

Then $\alpha_0(\mathbb{X}) = 2\mathbb{X}$, "even numbers" and $\alpha_1(\mathbb{X}) = 2\mathbb{X} + 1$, "odd numbers".

T first partitions into even and odd, then modular four classes, then modular eight, modular sixteen, etc.



CONVERGENCE????

Multiresolution analysis

$\Omega = [0, 1]$ with $w_0(x) = x/2$ and $w_1(x) = x/2 + 1/2$.

$\mathbb{X} = L^2(\mathbb{R})$ and α_0 be the "low pass filter" and α_1 be the "high pass filter" from a two-scale MRA associated with a wavelet basis.

Then $\alpha_0(\mathbb{X}) + \alpha_1(\mathbb{X}) = \mathbb{X}$ and they generate a recursive partition of $L^2(\mathbb{R})$ which is associated with the wavelet basis.

The resulting "multimeasure" is a subspace-valued measure.

Thank you for listening!

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Thanks!

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Questions?